# 100マス積分解答例

氷上優希 (@coolplus099)

平成 24 年 12 月 17 日

このファイルは「初等函数百升積分」(http://teruteru.bakufu.org/100integration.htm) の解答例です. 何度か確認はしましたが, 結局は有志の突貫制作ですのでミスなどありましたら連絡して下さると嬉しいです. また特殊な関数も計算上出てきますが詳しいことは記載できなかったので, もし補足等あれば併せてお願いします.

挑んでみた感想としては, 部分分数分解が煩雑で苦労しました. 後半の問題はほぼそんな感じです. テクニカルな問題はあまりありませんので純粋にいい練習問題ではないかなと思います. 一度はやってみてはいかかでしょうか.

便宜上, 以下のように番号を振ります.

| | $\int x$ | $\int x^2$ | $\int \sqrt{x}$ | $\int \frac{1}{x}$ | $\int \sin x$ | $\int \cos x$ | $\int e^x$ | $\int \log x$ | $\int \frac{1}{x^2+1}$ | $\int \frac{1}{x^2-1}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $x\ \mathrm{d}x$ | 1-1 | (1-2) | (1-3) | (1-4) | (1-5) | (1-6) | (1-7) | (1-8) | (1-9) | (1-10) |
| $x^2\ \mathrm{d}x$ | 1-2 | 2-2 | (2-3) | (2-4) | (2-5) | (2-6) | (2-7) | (2-8) | (2-9) | (2-10) |
| $\sqrt{x}\ \mathrm{d}x$ | 1-3 | 2-3 | (2-4) | (3-4) | (3-5) | (3-6) | (3-7) | (3-8) | (3-9) | (3-10) |
| $\frac{\mathrm{d}x}{x}$ | 1-4 | 2-4 | 3-4 | 4-4 | (4-5) | (4-6) | (4-7) | (4-8) | (4-9) | (4-10) |
| $\sin x\ \mathrm{d}x$ | 1-5 | 2-5 | 3-5 | 4-5 | 5-5 | (5-6) | (5-7) | (5-8) | (5-9) | (5-10) |
| $\cos x\ \mathrm{d}x$ | 1-6 | 2-6 | 3-6 | 4-6 | 5-6 | 6-6 | (6-7) | (6-8) | (6-9) | (6-10) |
| $e^x\ \mathrm{d}x$ | 1-7 | 2-7 | 3-7 | 4-7 | 5-7 | 6-7 | 7-7 | (7-8) | (7-9) | (7-10) |
| $\log x\ \mathrm{d}x$ | 1-8 | 2-8 | 3-8 | 4-8 | 5-8 | 6-8 | 7-8 | 8-8 | (8-9) | (8-10) |
| $\frac{\mathrm{d}x}{x^2+1}$ | 1-9 | 2-9 | 3-9 | 4-9 | 5-9 | 6-9 | 7-9 | 8-9 | 9-9 | (9-10) |
| $\frac{\mathrm{d}x}{x^2-1}$ | 1-10 | 2-10 | 3-10 | 4-10 | 5-10 | 6-10 | 7-10 | 8-10 | 9-10 | 10-10 |

以下,$C$ を積分定数とします.
問題番号に † が振ってあるものは特殊な関数が含まれているもので, (知識が無いので) Wolfram Alpha で出した解答のみ掲載したものです.

<u>1-1</u>

$$\int x^2\ \mathrm{d}x = \frac{x^3}{3} + C$$

<u>1-2</u>

$$\int x^3\ \mathrm{d}x = \frac{x^4}{4} + C$$

<u>1-3</u>

$$\int x\sqrt{x}\ \mathrm{d}x = \frac{2}{5}x^2\sqrt{x} + C$$

<u>1-4</u>

$$\int\ \mathrm{d}x = x + C$$

1-5

$$\begin{aligned}\int x\sin x\ \mathrm{d}x &= \int x(-\cos x)'\ \mathrm{d}x \\ &= -x\cos x+\int \cos x\mathrm{d}x \\ &= -x\cos x+\sin x+C\end{aligned}$$

1-6

$$\begin{aligned}\int x\cos x\ \mathrm{d}x &= \int x(\sin x)'\ \mathrm{d}x \\ &= x\sin x-\int \sin x\mathrm{d}x \\ &= x\sin x+\cos x+C\end{aligned}$$

1-7

$$\begin{aligned}\int xe^x\ \mathrm{d}x &= \int x(e^x)'\ \mathrm{d}x \\ &= xe^x-\int e^x\mathrm{d}x \\ &= xe^x-e^x+C\end{aligned}$$

1-8

$$\begin{aligned}\int x\log x\ \mathrm{d}x &= \int (\frac{1}{2}x^2)'\log x\ \mathrm{d}x \\ &= \frac{1}{2}x^2\log x-\frac{1}{2}\int x\ \mathrm{d}x \\ &= \frac{1}{2}x^2\log x-\frac{1}{4}x^2+C\end{aligned}$$

1-9

$$\begin{aligned}\int \frac{x}{x^2+1}\ \mathrm{d}x &= \frac{1}{2}\int \frac{(x^2+1)'}{x^2+1}\ \mathrm{d}x \\ &= \frac{1}{2}\log(x^2+1)+C\end{aligned}$$

1-10

1-9 と同様に,

$$\int \frac{x}{x^2-1}\ \mathrm{d}x = \frac{1}{2}\log(x^2-1)+C$$

2-2

$$\int x^4\ \mathrm{d}x = \frac{x^5}{5}+C$$

2-3

$$\int x^2\sqrt{x}\ \mathrm{d}x = \frac{2}{7}x^3\sqrt{x}+C$$

2-4

$$\int x \ \mathrm{d}x = \frac{1}{2}x^2 + C$$

2-5

$$\begin{aligned}\int x^2 \sin x \ \mathrm{d}x &= \int x^2(-\cos x)' \ \mathrm{d}x \\ &= -x^2\cos x + 2\int x\cos x \mathrm{d}x \\ &= -x^2\cos x + 2x\sin x + 2\cos x + C\end{aligned}$$

2-6

$$\begin{aligned}\int x^2 \cos x \ \mathrm{d}x &= \int x^2(\sin x)' \ \mathrm{d}x \\ &= x^2\sin x + 2\int x\sin x \mathrm{d}x \\ &= x^2\sin x + 2x\cos x - 2\sin x + C\end{aligned}$$

2-7

$$\begin{aligned}\int x^2 e^x \ \mathrm{d}x &= \int x^2(e^x)' \ \mathrm{d}x \\ &= x^2 e^x - 2\int x e^x \mathrm{d}x \\ &= x^2 e^x - 2xe^x + 2e^x + C\end{aligned}$$

2-8

$$\begin{aligned}\int x^2 \log x \ \mathrm{d}x &= \int (\frac{1}{3}x^3)' \log x \ \mathrm{d}x \\ &= \frac{1}{3}x^3\log x - \frac{1}{3}\int x^2 \ \mathrm{d}x \\ &= \frac{1}{3}x^3\log x - \frac{1}{9}x^3 + C\end{aligned}$$

2-9

$$\begin{aligned}\int \frac{x^2}{x^2+1} \ \mathrm{d}x &= \int \left(1 - \frac{1}{x^2+1}\right) \ \mathrm{d}x \\ &= x - \arctan x + C\end{aligned}$$

2-10

$$\begin{aligned}\int \frac{x^2}{x^2-1} \ \mathrm{d}x &= \int \left(1 - \frac{1}{x^2-1}\right) \ \mathrm{d}x \\ &= x - \int \frac{1}{x^2-1} \ \mathrm{d}x \\ &= x - \frac{1}{2}\int \left(\frac{1}{x-1} - \frac{1}{x+1}\right) \ \mathrm{d}x \\ &= x - \frac{1}{2}(\log|x-1| - \log|x+1|) + C\end{aligned}$$

3-4

$$\int \frac{1}{\sqrt{x}}\ \mathrm{d}x = 2\sqrt{x} + C$$

3-5†

$$\int \sqrt{x}\sin x\ \mathrm{d}x = -\sqrt{x}\cos x + \sqrt{\frac{\pi}{2}}\mathrm{C}\left(\sqrt{\frac{2}{\pi}x}\right) + C$$

$\mathrm{C}(x)$ は Fresnel 積分.

3-6†

$$\int \sqrt{x}\cos x\ \mathrm{d}x = \sqrt{x}\sin x - \sqrt{\frac{\pi}{2}}\mathrm{S}\left(\sqrt{\frac{2}{\pi}x}\right) + C$$

$\mathrm{S}(x)$ は Fresnel 積分.

3-7†

$$\int \sqrt{x}e^x\ \mathrm{d}x = \sqrt{x}e^x - \frac{\sqrt{\pi}}{2}\mathrm{erf}(i\sqrt{x}) + C$$

$\mathrm{erf}(x)$ は誤差関数, $i$ は虚数単位.

3-8

$$\begin{aligned}\int \sqrt{x}\log x\ \mathrm{d}x &= \int (\frac{2}{3}x^{\frac{3}{2}})' \log x\ \mathrm{d}x \\ &= \frac{2}{3}x^{\frac{3}{2}}\log x - \frac{2}{3}\int \sqrt{x}\ \mathrm{d}x \\ &= \frac{2}{3}x^{\frac{3}{2}}\log x - \frac{4}{9}x^{\frac{3}{2}} + C\end{aligned}$$

3-9

$t = \sqrt{x}$ とおく. このとき $\mathrm{d}t = \frac{1}{2\sqrt{x}}\mathrm{d}x$ であるから,

$$\int \frac{\sqrt{x}}{x^2+1}\ \mathrm{d}x = 2\int \frac{t^2}{t^4+1}\ \mathrm{d}t$$

$t^4+1 = (t^2+\sqrt{2}t+1)(t^2-\sqrt{2}t+1)$ を利用して部分分数分解を行い,

$$\begin{aligned}&= -\frac{1}{\sqrt{2}}\int \frac{t}{t^2+\sqrt{2}t+1}\ \mathrm{d}t + \frac{1}{\sqrt{2}}\int \frac{t}{t^2-\sqrt{2}t+1}\ \mathrm{d}t \\ &= -\frac{1}{2\sqrt{2}}\int \frac{(2t+\sqrt{2})-\sqrt{2}}{t^2+\sqrt{2}t+1}\ \mathrm{d}t + \frac{1}{2\sqrt{2}}\int \frac{(2t-\sqrt{2})+\sqrt{2}}{t^2-\sqrt{2}t+1}\ \mathrm{d}t \\ &= -\frac{1}{2\sqrt{2}}\int \frac{2t+\sqrt{2}}{t^2+\sqrt{2}t+1}\ \mathrm{d}t + \frac{1}{2\sqrt{2}}\int \frac{\sqrt{2}}{t^2+\sqrt{2}t+1}\ \mathrm{d}t + \frac{1}{2\sqrt{2}}\int \frac{2t-\sqrt{2}}{t^2-\sqrt{2}t+1}\ \mathrm{d}t + \frac{1}{2\sqrt{2}}\int \frac{\sqrt{2}}{t^2-\sqrt{2}t+1}\ \mathrm{d}t\end{aligned}$$

$$(\text{第 1 項}) = -\frac{1}{2\sqrt{2}}\int \frac{(t^2+\sqrt{2}t+1)'}{t^2+\sqrt{2}t+1}\ \mathrm{d}t = -\frac{1}{2\sqrt{2}}\log(t^2+\sqrt{2}t+1)$$

同様にして,

$$(\text{第 3 項}) = \frac{1}{2\sqrt{2}}\log(t^2-\sqrt{2}t+1)$$

また,

$$(\text{第 2 項}) = \frac{1}{2}\int \frac{\mathrm{d}t}{(t+\frac{1}{\sqrt{2}})^2+\frac{1}{2}}$$

$s = t + \frac{1}{\sqrt{2}}$ とおくと $\mathrm{d}s = \mathrm{d}t$ であるから,

$$\begin{aligned}
&= \frac{1}{2}\int \frac{\mathrm{d}s}{s^2 + \frac{1}{2}} \\
&= \frac{1}{\sqrt{2}}\int \frac{\sqrt{2}\mathrm{d}s}{(\sqrt{2}s)^2 + 1} \\
&= \frac{1}{\sqrt{2}}\arctan(\sqrt{2}s) = \frac{1}{\sqrt{2}}\arctan(\sqrt{2}t + 1)
\end{aligned}$$

同様にして,

$$(\text{第 4 項}) = \frac{1}{\sqrt{2}}\arctan(\sqrt{2}t - 1)$$

したがって,

$$\begin{aligned}
&\int \frac{\sqrt{x}}{x^2+1}\ \mathrm{d}x \\
= &\ \frac{1}{2\sqrt{2}}\Big(-\log(t^2 + \sqrt{2}t + 1) + \log(t^2 - \sqrt{2}t + 1) + 2\arctan(\sqrt{2}t + 1) + 2\arctan(\sqrt{2}t - 1)\Big) + C \\
= &\ \frac{1}{2\sqrt{2}}\Big(-\log(x + \sqrt{2x} + 1) + \log(x - \sqrt{2x} + 1) + 2\arctan(\sqrt{2x} + 1) + 2\arctan(\sqrt{2x} - 1)\Big) + C
\end{aligned}$$

<u>3-10</u>

$t = \sqrt{x}$ とおく．このとき $\mathrm{d}t = \frac{1}{2\sqrt{x}}\mathrm{d}x$ であるから,

$$\begin{aligned}
\int \frac{\sqrt{x}}{x^2-1}\ \mathrm{d}x &= \int \frac{\sqrt{t}^2}{x^4-1}\ \mathrm{d}x = \int \frac{\sqrt{t}^2}{(t-1)(t+1)(t^2+1)}\ \mathrm{d}x \\
&= \frac{1}{2}\left(\int \frac{\mathrm{d}x}{t-1} + \int \frac{\mathrm{d}x}{t+1} + \int \frac{2\mathrm{d}x}{t^2+1}\right) + C \\
&= \frac{1}{2}(\log|t-1| - \log|t+1|) + \arctan t + C \\
&= \frac{1}{2}(\log|\sqrt{x}-1| - \log|\sqrt{x}+1|) + \arctan\sqrt{x} + C
\end{aligned}$$

<u>4-4</u>

$$\int \frac{1}{x^2}\ \mathrm{d}x = -\frac{1}{x} + C$$

<u>4-5</u>$^\dagger$

$$\int \frac{\sin x}{x}\ \mathrm{d}x = \mathrm{Si}(x) + C$$

$\mathrm{Si}(x)$ は正弦積分.

<u>4-6</u>$^\dagger$

$$\int \frac{\cos x}{x}\ \mathrm{d}x = \mathrm{Ci}(x) + C$$

$\mathrm{Ci}(x)$ は余弦積分.

<u>4-7</u>$^\dagger$

$$\int \frac{e^x}{x}\ \mathrm{d}x = \mathrm{Ei}(x) + C$$

$\mathrm{Ei}(x)$ は指数積分.

4-8

$$\int \frac{\log x}{x}\ \mathrm{d}x = \int \log x (\log x)'\ \mathrm{d}x = \frac{1}{2}(\log x)^2$$

4-9

$$\begin{aligned}\int \frac{\mathrm{d}x}{x(x^2+1)} &= \int \left(\frac{1}{x} - \frac{x}{x^2+1}\right)\ \mathrm{d}x \\ &= \log x - \frac{1}{2}\int \frac{(x^2+1)'}{x^2+1}\ \mathrm{d}x \\ &= \log x - \frac{1}{2}\log(x^2+1) + C\end{aligned}$$

4-10

$$\begin{aligned}\int \frac{\mathrm{d}x}{x(x^2-1)} &= \frac{1}{2}\int \left(-\frac{2}{x} + \frac{1}{x-1} + \frac{1}{x+1}\right)\ \mathrm{d}x \\ &= \frac{1}{2}\left(-2\log|x| + \log|x-1| + \log|x+1|\right) + C\end{aligned}$$

5-5

$$\begin{aligned}\int \sin^2 x \mathrm{d}x &= \frac{1}{2}\int (1-\cos 2x)\ \mathrm{d}x \\ &= \frac{1}{4}(2x - \sin 2x) + C\end{aligned}$$

5-6

$$\begin{aligned}\int \sin x \cos x \mathrm{d}x &= \frac{1}{2}\int \sin 2x\ \mathrm{d}x \\ &= -\frac{1}{4}\cos 2x + C\end{aligned}$$

5-7, 6-7

$$\begin{aligned}\int e^x \sin x \mathrm{d}x &= \int (e^x)' \sin x \mathrm{d}x \\ &= e^x \sin x - \int e^x \cos x \mathrm{d}x \\ \int e^x \cos x \mathrm{d}x &= \int (e^x)' \cos x \mathrm{d}x \\ &= e^x \cos x + \int e^x \sin x \mathrm{d}x\end{aligned}$$

上の 2 式を連立して解くと,

$$\begin{aligned}\int e^x \sin x \mathrm{d}x &= \frac{1}{2}e^x(\sin x - \cos x) + C \\ \int e^x \cos x \mathrm{d}x &= \frac{1}{2}e^x(\sin x + \cos x) + C\end{aligned}$$

5-8†

$$\int \log x \sin x \mathrm{d}x = -\log x \cos x + \mathrm{Ci}(x) + C$$

$\mathrm{Ci}(x)$ は余弦積分.

5-9†

$$\int \frac{\sin x}{x^2+1}\ \mathrm{d}x = \frac{(e^2-1)(\mathrm{Ci}(i-x)+\mathrm{Ci}(i+x)) + i(1+e^2)(\mathrm{Si}(i-x)+\mathrm{Si}(i+x))}{4e} + C$$

$\mathrm{Si}(x)$ は正弦積分, $\mathrm{Ci}(x)$ は余弦積分, $i$ は虚数単位.

5-10†

$$\int \frac{\sin x}{x^2-1}\ \mathrm{d}x = \frac{\sin(1)(\mathrm{Ci}(1-x)+\mathrm{Ci}(1+x)) + \cos(1)(\mathrm{Si}(1-x)+\mathrm{Si}(1+x))}{2} + C$$

$\mathrm{Si}(x)$ は正弦積分, $\mathrm{Ci}(x)$ は余弦積分.

6-6

$$\begin{aligned}\int \cos^2 x \mathrm{d}x &= \frac{1}{2}\int (1+\cos 2x)\ \mathrm{d}x \\ &= \frac{1}{4}(2x+\sin 2x) + C\end{aligned}$$

6-8†

$$\int \log x \cos x \mathrm{d}x = \log x \sin x - \mathrm{Si}(x) + C$$

$\mathrm{Si}(x)$ は正弦積分.

6-9†

$$\int \frac{\cos x}{x^2+1}\ \mathrm{d}x = \frac{(e^2-1)(\mathrm{Si}(i-x)+\mathrm{Si}(i+x)) - i(1+e^2)(\mathrm{Ci}(i-x)-\mathrm{Ci}(i+x))}{4e} + C$$

$\mathrm{Si}(x)$ は正弦積分, $\mathrm{Ci}(x)$ は余弦積分, $i$ は虚数単位.

6-10†

$$\int \frac{\cos x}{x^2-1}\ \mathrm{d}x = \frac{\cos(1)(\mathrm{Ci}(1-x)+\mathrm{Ci}(1+x)) + \sin(1)(\mathrm{Si}(1-x)+\mathrm{Si}(1+x))}{2} + C$$

$\mathrm{Si}(x)$ は正弦積分, $\mathrm{Ci}(x)$ は余弦積分.

7-7

$$\int e^{2x}\ \mathrm{d}x = \frac{1}{2}e^{2x} + C$$

7-8†

$$\int e^x \log x\ \mathrm{d}x = e^x \log x \sin x - \mathrm{Ei}(x) + C$$

$\mathrm{Ei}(x)$ は指数積分.

7-9†

$$\int \frac{e^x}{x^2+1}\ \mathrm{d}x = -\frac{1}{2}ie^{-i}(e^{2i}\mathrm{Ei}(x-i) - \mathrm{Ei}(x+i)) + C$$

$\mathrm{Ei}(x)$ は指数積分, $i$ は虚数単位.

7-10†

$$\int \frac{e^x}{x^2-1}\ \mathrm{d}x = \frac{1}{2}\left(e\mathrm{Ei}(x-1) - \frac{1}{e}\mathrm{Ei}(x+1)\right) + C$$

$\mathrm{Ei}(x)$ は指数積分.

8-8

$$\begin{aligned}\int(\log x)^2\ \mathrm{d}x &= \int(x)'(\log x)^2\ \mathrm{d}x\\ &= x(\log x)^2-2\int\log x\ \mathrm{d}x\\ &= x((\log x)^2-2\log x+2)+C\end{aligned}$$

8-9†

$$\int\frac{\log x}{x^2+1}\ \mathrm{d}x=\frac{1}{2}i(-\mathrm{Li}_2(-ix)+\mathrm{Li}_2(ix)+(\log(1-ix)-\log(1+ix))\log x)+C$$

$\mathrm{Li}_2(x)$ は多重対数関数, $i$ は虚数単位.

8-10†

$$\int\frac{\log x}{x^2-1}\ \mathrm{d}x=\frac{1}{2}\left(-\mathrm{Li}_2(1-x)-\mathrm{Li}_2(-x)-\log x\log(x+1)\right)+C$$

$\mathrm{Li}_2(x)$ は多重対数関数.

9-9

$x=\tan t$ とおく．このとき $\mathrm{d}x=\frac{1}{\cos^2 t}\mathrm{d}t=(1+\tan^2 t)\mathrm{d}t$ であるから,

$$\begin{aligned}\int\frac{1}{(x^2+1)^2}\ \mathrm{d}x &= \int\frac{\mathrm{d}t}{1+\tan^2 t}\\ &= \int\cos^2 t\mathrm{d}t\\ &= \frac{1}{4}(2t+\sin 2t)+C\end{aligned}$$

$u=\tan\frac{\theta}{2}$ とおくと $\sin u=\frac{2u}{u^2+1}$ であるから,

$$\begin{aligned} &= \frac{1}{4}\left(2\arctan x+\frac{2x}{x^2+1}\right)+C\\ &= \frac{1}{2}\left(\arctan x+\frac{x}{x^2+1}\right)+C\end{aligned}$$

9-10

$$\begin{aligned}\int\frac{\mathrm{d}x}{(x^2-1)(x^2+1)} &= -\frac{1}{4}\int\left(\frac{2}{x^2+1}-\frac{1}{x-1}+\frac{1}{x+1}\right)\ \mathrm{d}x\\ &= -\frac{1}{4}(2\arctan x-\log|x-1|+\log|x+1|)+C\end{aligned}$$

10-10

$$\begin{aligned}\int\frac{\mathrm{d}x}{(x^2-1)^2} &= \frac{1}{4}\int\left(-\frac{1}{x-1}+\frac{1}{(x-1)^2}+\frac{1}{x+1}+\frac{1}{(x+1)^2}\right)\ \mathrm{d}x\\ &= \frac{1}{4}\left(-\log|x-1|-\frac{1}{x-1}+\log|x+1|-\frac{1}{x+1}\right)+C\\ &= -\frac{1}{4}\left(\log|x-1|-\log|x+1|+\frac{2x}{x^2-1}\right)+C\end{aligned}$$

以上.